FLOW OF A VISCOUS FILM OVER THE SURFACE OF AN INVISCID FLUID

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1. Equilibrium Width of a Film. If a certain amount of a light fluid is placed on an unconfined surface of a heavier one, while the fluids do not mix the final state of the system is determined by the sign if the quantity γ:

$$\gamma = \gamma_1 + \gamma_a - \gamma_{1a}, \tag{1.1}$$

where γ_1, γ_a, and γ_{1a} are the surface tension coefficients of the light fluid—heavy fluid, light fluid—atmosphere, and heavy fluid—atmosphere boundaries, respectively (Fig. 1). If $\gamma < 0$ (the quantity γ is called the spread coefficient [1]), a monomolecular film is formed. For $\gamma > 0$ a film of finite width H_0 is formed, with the width determined by the equation [2]

$$H_0^2 = \frac{2\gamma}{\rho g(1-\rho/\rho_1)}, \tag{1.2}$$

where ρ and ρ_1 are the densities of the upper and lower fluids, and g is the acceleration.

The total force acting on the film cross section AB from the right equals the sum of surface tensions of the horizontal boundaries of the upper fluid due to hydrostatic pressure of its layer of height H. The force acting on AB from the left equals the surface tension of the free boundary of the lower fluid due to the pressure of its layer of height H_1. Equating these forces, we have

$$\gamma_1 + \gamma_a - \rho g H^2/2 = \gamma_{1a} - \rho_1 g H_1^2/2. \tag{1.3}$$

Taking into account the Archimedes law $H_1 = (\rho/\rho_1)H$, $H_1 + H_2 = H$, we obtain (1.2) from (1.3). Considering the problem of equilibrium width of a film by the ordinary methods of hydrostatics, discussed, e.g., in [3], it can be shown by account of the boundary conditions at point O that relation (1.3) is an exact consequence of the equations of hydrostatics. Not being interested in the shape of the film at distances of order H_0 from the boundary, it can be assumed that the action of surface forces of all boundaries reduces to the action of a force of magnitude γ per unit length on the film boundary. This approximation, correct in the static case, is also valid in some flow region under the action of forces of the same order as γ.

The interest in this group of problems has increased with the appearance of a new method of producing high-quality glass plates, the float process [4]. At one stage of this process the flow of molten glass flows over into a bath with molten tin, while the edges of the glass strip remain free. After the time necessary for the glass surface to remain smooth, its width reaches an equilibrium value H_0 equal to 6-7 mm. In this example we encounter a flow of a viscous film over a surface of an inviscid fluid, since the viscosity of tin is negligibly low at high temperatures in comparison with the viscosity of glass, and there are no tangential stresses on their common boundary. All viscous effects in this type of flow are due to internal friction due to the change in shape of the film itself.

2. Derivation of Equations of Motion. We start from the equation of continuity and the equations for the horizontal components of the momentum for an incompressible fluid of constant density:

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0; \tag{2.1}$$

$$\rho(\partial u/\partial t + \partial u^2/\partial x + \partial uv/\partial y + \partial uw/\partial z) = \partial\sigma_{xx}/\partial x + \partial\sigma_{xy}/\partial y + \partial\sigma_{xz}/\partial z; \tag{2.2}$$

$$\rho(\partial v/\partial t + \partial uv/\partial x + \partial v^2/\partial y + \partial vw/\partial z) = \partial\sigma_{yx}/\partial x + \partial\sigma_{yy}/\partial y + \partial\sigma_{yz}/\partial z, \tag{2.3}$$

where u, v, w are the velocity components along the x, y, z axes, and σ_{xx}, σ_{xy}... are the components of the stress tensor. The positions of the upper and lower surfaces of the film $H_2(t, x, y)$ and $-H_1(t, x, y)$ are assumed to be slowly varying functions of x and y, so that the

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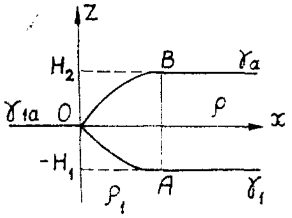

Fig. 1

characteristic intervals of their variation are large in comparison with the width $H = H_1 + H_2$, and the inequalities $(\partial H_i/\partial x)^2 \ll 1$, $(\partial H_i/\partial y)^2 \ll 1$, i = 1, 2 are also valid. In integrating Eqs. (2.1)-(2.3) over z we use the equation

$$\int_{-H_1}^{H_2} \frac{\partial f}{\partial t}(t, x, y, z)\, dz = \frac{\partial}{\partial t}\int_{-H_1}^{H_2} f dz - f\Big|_{-H_1}\frac{\partial H_1}{\partial t} - f\Big|_{H_2}\frac{\partial H_2}{\partial t} \tag{2.4}$$

and similar equations for differentiation with respect to x and y. We introduce averages of the velocity and the stresses over the width

$$\langle u\rangle = \frac{1}{H}\int_{-H_1}^{H_2} u dz, \quad \langle\sigma_{xx}\rangle = \frac{1}{H}\int_{-H_1}^{H_2}\sigma_{xx} dz, \ldots$$

The following kinematic conditions are valid for w values on the surfaces

$$w\,|_{H_2} = \left(\frac{\partial H_2}{\partial t} + u\frac{\partial H_2}{\partial x} + v\frac{\partial H_2}{\partial y}\right)_{H_2}, \quad w\,|_{-H_1} = -\left(\frac{\partial H_1}{\partial t} + u\frac{\partial H_1}{\partial x} + v\frac{\partial H_1}{\partial y}\right)_{-H_1}. \tag{2.5}$$

Integration of (2.1) over z with account of (2.4), (2.5) gives an equation of continuity in the form

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\langle u\rangle) + \frac{\partial}{\partial y}(H\langle v\rangle) = 0.$$

The velocity components u and v can be written in the form

$$u = \langle u\rangle + \delta u,\ v = \langle v\rangle + \delta v,\ \langle\delta u\rangle = \langle\delta v\rangle = 0.$$

We assume that the horizontal velocities vary little over the film width, i.e., $|\delta u| \ll |\langle u\rangle|$, $|\delta v| \ll |\langle v\rangle|$. After integration over z the left-hand side of Eq. (2.2) can be written in the form

$$\rho[\partial H\langle u\rangle/\partial t + \partial H\langle u\rangle^2/\partial x + \partial H\langle u\rangle\langle v\rangle/\partial y + O_2],$$

where O_2 is the sum of second-order terms. Integration of the right-hand side of (2.2) gives the expression

$$\frac{\partial H\langle\sigma_{xx}\rangle}{\partial x} + \frac{\partial H\langle\sigma_{xy}\rangle}{\partial y} + \left[\sigma_{xz} - \sigma_{xx}\frac{\partial H_2}{\partial x} - \sigma_{xy}\frac{\partial H_2}{\partial y}\right]_{H_2} - \left[\sigma_{xz} + \sigma_{xx}\frac{\partial H_1}{\partial x} + \sigma_{xy}\frac{\partial H_1}{\partial y}\right]_{-H_1}. \tag{2.6}$$

The components of the outer normals to the surfaces $-H_1$ and H_2 can be written approximately in the form

$$\mathbf{n}_1 = (-\partial H_1/\partial x,\ -\partial H_1/\partial y,\ -1),\ \mathbf{n}_2 = (-\partial H_2/\partial x,\ -\partial H_2/\partial y,\ 1).$$

It is now seen that the square brackets of (2.6) contain the x-components of the forces acting on the film surface. We assume that forces are absent at the upper boundary of the film, but at the lower boundary

$$\left[\sigma_{xz} + \sigma_{xx}\frac{\partial H_1}{\partial x} + \sigma_{xy}\frac{\partial H_1}{\partial y}\right]_{-H_1} = \left[\sigma^1_{xz} + \sigma^1_{xx}\frac{\partial H_1}{\partial x} + \sigma^1_{xy}\frac{\partial H_1}{\partial y}\right]_{-H_1},$$

where σ^1 is the stress tensor in the lower fluid.

The vanishing of the z-component of the force at the upper boundary is written, up to first-order quantities, in the form

$$\sigma_{zz}|_{H_2} = 0. \tag{2.7}$$

For a Newtonian fluid

$$\sigma_{xx} = -p + 2\mu\partial u/\partial x,\ \sigma_{xy} = \mu(\partial u/\partial y + \partial v/\partial x),\quad \sigma_{zz} = -p + 2\mu\partial w/\partial z, \tag{2.8}$$

where p is the pressure and μ is the viscosity coefficient. Taking into account (2.1), condition (2.7) gives

$$p|_{H_2} = -2\mu\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]_{H_2} = -2\mu\left(\frac{\partial\langle u\rangle}{\partial x} + \frac{\partial\langle v\rangle}{\partial y}\right) - 2\mu\left(\frac{\partial\delta u}{\partial x} + \frac{\partial\delta v}{\partial y}\right)_{H_2}. \tag{2.9}$$

Bearing in mind that δu and δv are not only small, but are also slowly varying functions, i.e., differentiation with respect to x and y enhances the order of smallness of the last term in (2.9), it can be neglected within the accuracy assumed by us. In that case

$$p|_{H_2} = -2\mu(\partial\langle u\rangle/\partial x + \partial\langle v\rangle/\partial y). \tag{2.10}$$

We note that the transition from (2.9) to (2.10) is essentially equivalent to the assumption of linear dependence of the velocity w on z, since the equality w = A(t, x, y)z + B(t, x, y) follows from the equation

$$\partial w/\partial z = -(\partial\langle u\rangle/\partial x + \partial\langle v\rangle/\partial y) \tag{2.11}$$

We assume that the pressure in the film varies hydrodynamically: $\partial p/\partial z = -\rho g$.

Then from (2.10) we obtain

$$p = \rho g(H_2 - z) - 2\mu(\partial\langle u\rangle/\partial x + \partial\langle v\rangle/\partial y).$$

Using (2.8), (2.11), we have on the lower boundary

$$\sigma_{zz}|_{-H_1} = -\rho g H. \tag{2.12}$$

The representation $\sigma^1_{ik} = -p^1\delta_{ik}$, where δ_{ik} is the Kronecker symbol.

With varying degree of accuracy one can take into account the flow properties of the lower fluid, choosing various expressions for the pressure p^1. The simplest property, suitable for low-intensity flows, is the assumption that the pressure in it is hydrostatic. Using (2.12), from the condition $\sigma_{zz}|_{-H_1} = \sigma^1_{zz}|_{-H_1}$, we then obtain $H_1 = (\rho/\rho_2)H$, i.e., in our approximation the condition of hydrostatic equilibrium is satisfied locally. One can now write

$$H\langle\sigma_{xx}\rangle = -\rho g\frac{H^2}{2} + 4\mu H\frac{\partial\langle u\rangle}{\partial x} + 2\mu H\frac{\partial\langle v\rangle}{\partial y} + O_2,\quad H\langle\sigma_{xy}\rangle = \mu H\left(\frac{\partial\langle u\rangle}{\partial y} + \frac{\partial\langle v\rangle}{\partial x}\right) + O_2.$$

Besides,

$$\left[\sigma^1_{xz} + \sigma^1_{xx}\frac{\partial H_1}{\partial x} + \sigma^1_{xy}\frac{\partial H_1}{\partial y}\right]_{-H_1} = -\rho_1 g H_1\frac{\partial H_1}{\partial x} = -\frac{\rho^2}{2\rho_1}\frac{\partial H^2}{\partial x}.$$

Collecting the expressions obtained, turning to substantial description of the inertial terms, discarding the averaging sign, and using the fact that the equation for the y-component of the momentum is obtained from the equation for the x-component by replacing $x \rightleftarrows y$, $u \rightleftarrows v$, we have the following system for the functions H, u, v of the variables t, x, y:

$$\partial H/\partial t + \partial Hu/\partial x + \partial Hv/\partial y = 0; \tag{2.13}$$

$$\rho H\frac{du}{dt} = \frac{\partial}{\partial x}\left[-\rho(1-\rho/\rho_1)g\frac{H^2}{2} + 4\mu H\frac{\partial u}{\partial x} + 2\mu H\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial y}\left[\mu H\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]; \tag{2.14}$$

$$\rho H\frac{dv}{dt} = \frac{\partial}{\partial y}\left[-\rho(1-\rho/\rho_1)g\frac{H^2}{2} + 4\mu H\frac{\partial v}{\partial y} + 2\mu H\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}\left[\mu H\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right], \tag{2.15}$$

where $d/dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y$.

3. General Formulation. We write Eqs. (2.14), (2.15) in the form

$$\rho H dv_i/dt = \partial S_{ik}/\partial x_k + F_i,\ i,\ k = 1,\ 2,$$

where by k we understand summation; S_{ik}, two-dimensional stress tensor; and F_i, vector of external forces. The tensor S_{ik} can be represented in the form

$$S_{ik} = -P\delta_{ik} + S'_{ik} = -P\delta_{ik} + \lambda HD\delta_{ik} + 2\mu He_{ik}.$$

where P is the two-dimensional pressure:

$$P = \rho(1 - \rho/\rho_1)gH^2/2; \tag{3.1}$$

e_{ik} is the velocity deformation tensor, and D is the two-dimensional velocity. The quantities λH and μH play the role of Lamé constants. Using the deviator d_{ik}, the viscous part of the stress tensor S'_{ik} can be written in the form

$$S'_{ik} = d_{ik} + \mu' HD\delta_{ik},\quad d_{ik} = 2\mu H\left(e_{ik} - \frac{1}{2}\delta_{ik}D\right). \tag{3.2}$$

Thus, the film flow can be described as flow of a two-dimensional fluid with the equation of state (3.1) and nonvanishing Lamé "constants" (which due to the factor H are functions of x_i and t) or the two-dimensional analog of the bulk viscosity μ'H, with λ = 2 , μ' = 3μ. This stress tensor should not be used if it is necessary to take into account the film extension.* In the case of planar flow (in the three-dimensional sense) for $v_2 = 0$ we have $S'_1 = 4\mu H\partial v_1/\partial x$.† In passing to the one-dimensional flow and averaging Eq. (2.14) over y we obtain $S'_1 = \mu' s\partial v_1/\partial x$, where s is the area of the transverse cross section. Thus, the quantity μ' = 3μ is practically the "longitudinal viscosity" introduced in [7].

The film motion over a layer of an inviscid fluid can be interpreted as slipping over a solid surface deflected by the weight of the film by the quantity H_1. For $\rho_1 = \infty$ we obtain a slipping of the film over a horizontal plane, with $P = \rho gH^2/2$. If we now put in the equations μ = λ = 0, we obtain the ordinary equations of "soft water," so that the given model can also be considered as a generalization of the "soft water" model.

In the present model we take into account only the inertia of the film itself, therefore it can used when inertial effects in the lower fluid have little effect on the film flow. Hence also follow the special forms of the fast film flows, such as the accelerated motion of a film of constant width in a direction parallel to the edge. If the film moves together with the fluid flow, these flows can also be considered within the model considered, introducing a moving coordinate system and including in F_i inertial forces.

It is advisable to give the formulation of the equations obtained in arbitrary coordinates. The general equation of motion of a continuous medium with account of the replacement of ρ, λ, μ by ρH, λH, μH is [8]

$$\rho Ha^i = \nabla_j S^{ij} + F^i,\ S^{ij} = -Pg^{ij} + \lambda HDg^{ij} + 2\mu He^{ij},\ i,\ j = 1,\ 2. \tag{3.3}$$

Here the superscripts are contravariant components of tensors, the subscripts are covariant components, g^{ij} is the metric tensor, a^i is the acceleration, ∇_j is the symbol of covariant differentiation, and $e_{ij} = (\nabla_i v_j + \nabla_j v_i)/2$. Changing insignificantly the calculations performed in [8], from (3.3) we obtain an equation of the form

$$\rho Ha^i = -\nabla^i P + (\lambda + \mu)H\nabla^i D + \mu H\Delta v^i + D\nabla^i(\lambda H) + e^{ij}\nabla_j(2\mu H), \tag{3.4}$$

where $\Delta = \nabla^j\nabla_j$ is the Laplace operator. The raising of subscripts is realized by contracting with the metric tensor, for example, $\nabla^i = g^{ij}\nabla_j$, $e^{ij} = g^{ik}g^{jl}e_{kl}$. In particular, in polar coordinates (r, radius: φ , angle) Eqs. (3.4), written fown for the physical components of vectors and tensors, acquire the form

$$\rho H\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\varphi}{r}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi^2}{r}\right) = F_r - \frac{\partial P}{\partial r} + (\lambda + \mu)H\frac{\partial D}{\partial r}$$
$$+ \mu H\left(\Delta v_r - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi} - \frac{v_r}{r^2}\right) + D\frac{\partial \lambda H}{\partial r} + 2\frac{\partial \mu H}{\partial r}e_{rr} + \frac{2}{r}\frac{\partial \mu H}{\partial \varphi}e_{r\varphi},$$

*The flow of viscous films with a stress tensor of shape (3.2) was treated in [5].
†A flow of this nature was considered in [6].

$$\rho H\left(\frac{\partial v_\varphi}{\partial t}+v_r\frac{\partial v_\varphi}{\partial r}+\frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial \varphi}+\frac{v_r v_\varphi}{r}\right)=F_\varphi-\frac{1}{r}\frac{\partial P}{\partial \varphi}+(\lambda+\mu)\frac{H}{r}\frac{\partial D}{\partial \varphi}+$$
$$+\mu H\left(\Delta v_\varphi+\frac{2}{r^2}\frac{\partial v_r}{\partial \varphi}-\frac{v_\varphi}{r^2}\right)+\frac{D}{r}\frac{\partial \lambda H}{\partial \varphi}+2\frac{\partial \mu H}{\partial r}e_{r\varphi}+\frac{2}{r}\frac{\partial \mu H}{\partial \varphi}e_{\varphi\varphi}, \qquad (3.5)$$

where

$$D=\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi};\quad \Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2};$$

$$e_{rr}=\frac{\partial v_r}{\partial r};\quad e_{r\varphi}=\frac{1}{2}\left(\frac{\partial v_\varphi}{\partial r}-\frac{v_\varphi}{r}+\frac{1}{r}\frac{\partial v_r}{\partial \varphi}\right);\quad e_{\varphi\varphi}=\frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi}+\frac{v_r}{r};$$

$P=\rho(1-\rho/\rho_1)gH^2/2$. The continuity equation acquires the form

$$\frac{\partial H}{\partial t}+v_r\frac{\partial H}{\partial r}+\frac{v_\varphi}{r}\frac{\partial H}{\partial \varphi}+HD=0. \qquad (3.6)$$

4. Examples. A. One-Dimensional Film Spreading. Let the variables H and u depend only on time t and the coordinate x, while v = 0. From (2.13), (2.14) we than obtain the system

$$\partial H/\partial t+u\partial H/\partial x+H\partial u/\partial x=0; \qquad (4.1)$$

$$H\frac{du}{dt}=\frac{\partial}{\partial x}\left[-(1-\rho/\rho_1)g\frac{H^2}{2}+4\nu H\frac{\partial u}{\partial x}\right], \qquad (4.2)$$

where ν is the kinematic viscosity coefficient. We assume a creeping flow under the action of the force γ (1.1), applied to the edge of the film. We note that the hydrostatic pressure force of the lower fluid acting on the edge is included in the equation of motion. Equating the right-hand side of (4.2) to zero and integrating the equation obtained, we have

$$(1-\rho/\rho_1)gH^2/2-4\nu H\partial u/\partial x=\gamma/\rho. \qquad (4.3)$$

Instead of x we introduce the Lagrangian coordinate a by the equations $x = x(t, a)$, $x(0, a) = a$. Denoting $x_a = \partial x/\partial a$, we then obtain $\partial u/\partial x = (\partial x_a/\partial t)/x_a$. Equation (4.1) gives the relation $Hx_a = H(0, a)$, hence $x_a = H(0, a)/H$ and, consequently, $\partial u/\partial x = -(\partial H/\partial t)/H$. Substituting this expression into (4.3), we obtain for $H(t, a)$ the equation

$$(1-\rho/\rho_1)g\frac{H^2}{2}+4\nu\frac{\partial H}{\partial t}=\gamma/\rho. \qquad (4.4)$$

Thus, transforming to the Lagrangian coordinate we have eliminated differentiation over the spatial variable and have obtained an ordinary differential equation. Introducing the dimensionless height h and the constant k by the equations

$$h=H/H_0,\ \varkappa=\rho g(1-\rho/\rho_1)H_0/8\mu, \qquad (4.5)$$

where H_0 is given by expression (1.2), we reduce (4.4) to the form

$$dh/dt+\varkappa(h^2-1)=0.$$

The equation is easily integrated

$$h(t,a)=1+\frac{2}{[1+2/(h(0,a)-1)]e^{2\varkappa t}-1}. \qquad (4.6)$$

This equation is also valid for $h(0, a) < 1$, when the film extends to a width less than H_0. The position of the element with coordinate a is given by the expression

$$x(t,a)=\int_0^a x_a da=\int_0^a \frac{h(0,a)}{h(t,a)}da.$$

For flows of molten glass over molten tin with $\rho = 2460$ kg/m³, $\rho_1 = 6450$ kg/m³, $\gamma_a = 0.267$ J/m², $\gamma_{1a} = 0.497$ J/m², $\gamma_1 = 0.392$ J/m², $\mu = 10^3$ kg/m·sec we obtain at temperature 1000°C a characteristic spreading time $1/2\varkappa = 1$ min, which agrees in order of magnitude with the experimental data of [4].

B. A Film of Thickness KH_0 Flowing from the Origin of Coordinates in the Direction of the x Axis with Velocity U. This problem is solved on the basis of the results obtained in

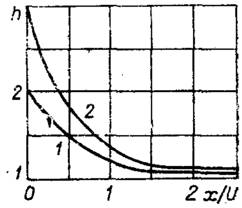

Fig. 2

the preceding example. For a film element flowing at moment t_0 the thickness at moment t is given by the modified Eq. (4.6):

$$h(t,\ t_0) = 1 + 2/\{\exp[2\varkappa(t - t_0)](K + 1)/(K - 1) - 1\}. \tag{4.7}$$

After time $t - t_0$ the element is extended by $K/h(t,\ t_0)$ times. Taking into account the length of all elements during the time interval $t_0 \leqslant \tau \leqslant t$, its position is determined by the expression

$$x(t,\ t_0) = \int\limits_{t_0}^{t} \frac{KUd\tau}{h(t,\tau)} = KU\left[(t - t_0) - \frac{1}{\varkappa}\ln\frac{2K}{K+1} + \frac{1}{\varkappa}\ln\left(1 + \frac{K-1}{K+1}e^{-2\varkappa(t-t_0)}\right)\right]. \tag{4.8}$$

With the lapse of a time somewhat larger than $1/2\varkappa$, a stationary flow regime is established, at which the film can be separated into two portions. At the first the width varies from K to 1, and in the second a film of unit width flows with velocity KU. In particular, the profile of its edge W is given by the equation

$$W(t) = x(t, 0) = KU\left(t - \frac{1}{\varkappa}\ln\frac{2K}{K+1}\right).$$

The second term in the brackets gives an edge retardation in comparison with the motion of a film edge of width H_0, flowing with velocity KU. The profile of initial portion h(x), which can be determined from (4.7), (4.8), is shown for the cases K = 2 and 3 on Fig. 2 (curves 1 and 2, respectively).

C. Radial Flows of Films with Account of a Linear Tension at the Edge. Along with the force γ, a tension force of the line of separation of the three media of magnitude γ_l/R is also acting on each unit length of the film edge, where R is the radius of curvature. We restrict ourselves to cases of creeping radial flows ($v_\varphi = 0$, $v_r = v$) of films of homogeneous thickness $H = H(t)$. Equations (3.5), (3.6) acquire then the form

$$\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) = 0; \tag{4.6a}$$

$$\partial H/\partial t + H(\partial v/\partial r + v/r) = 0. \tag{4.7a}$$

From (4.6a) we obtain $v = A(t)r + B(t)/r$. For a film in the form of a circle of radius R, a "lens," we obtain $v = A(t)r$ from the condition of finite velocity at r = 0. The radial component of the stress tensor, which is needed for the formulation of boundary conditions, is

$$S_{rr} = -P + \lambda H(\partial v/\partial r + v/r) + 2\mu H\partial v/\partial r.$$

Taking into account the relation $A = -(dH/dt)/2H$, following from (4.7a), the condition of equality of forces at the boundary gives the equation

$$3\mu\frac{dH}{dt} + \rho(1 - \rho/\rho_1)g\frac{H^2}{2} = \gamma + \gamma_l/R. \tag{4.8a}$$

Putting here $dH/dt = 0$, we obtain a new value of equilibrium width $H_* > H_0$, depending on R [2]:

$$H_*^2 = \frac{2(\gamma + \gamma_l/R)}{\rho g(1 - \rho/\rho_1)}.$$

The conservation condition of film volumes gives the relation

$$R^2(t)H(t) = R^2(0)H(0). \quad (4.9)$$

For the dimensionless height h(t) we obtain from (4.8a), (4.9) the equation

$$dh/dt + \varkappa_1(h^2 - 1) = \varepsilon\sqrt{h}, \quad (4.10)$$

where $\varkappa_1 = 4\varkappa/3$; $\varepsilon = \gamma_l/[3\mu H_0 R(0)\sqrt{h(0)}]$. For sufficiently large R(0) the quantity ε is small For $\varepsilon = 0$, when we neglect the linear tension, we obtain Eq. (4.6) for h(t) with replacing $\varkappa$ by $\varkappa_1$. For $\varepsilon \neq 0$ we have

$$t = \int_h^{h(0)} \frac{dh}{\varkappa_1(h^2 - 1) - \varepsilon\sqrt{h}}.$$

Knowing the film thickness, the edge position is found from (4.9).

Another case is that of an infinite film with a hole of radius R. Here v = B(t)/R, and it follows from (4.7a) that H = const. The conditions at the edge give

$$\rho(1 - \rho/\rho_1)gH^2/2 + 2\mu HB/R^2 = \gamma - \gamma_l/R. \quad (4.11)$$

Taking into account that dR/dt = B/R, from (4.11) we obtain the following equation for the motion of the edge:

$$dR/dt + \alpha R/2\mu H = -\gamma_l/2\mu H, \quad (4.12)$$

where $\alpha = \rho g(1 - \rho/\rho_1)H^2/2 - \gamma$. If $\alpha > 0$, i.e., $H > H_0$, equilibrium is impossible and the hole tightens. The dynamics of the process is given by the solution of Eq. (4.12):

$$R(t) = (R(0) + \gamma_l/\alpha)\exp(-\alpha t/2\mu H) - \gamma_l/\alpha. \quad (4.13)$$

For $\alpha = 0$, $H = H_0$ the hole also tightens:

$$R(t) = R(0) - \gamma_l t/2\mu H_0.$$

When $\alpha < 0$, $H < H_0$ an equilibrium state is possible, in which the width H_* and the radius R_* are related by

$$H_*^2 = \frac{2(\gamma - \gamma_l/R_*)}{\rho g(1 - \rho/\rho_1)}.$$

If R(0) and H(0) do not satisfy this condition, it is seen from (4.13) that $\delta R = R - R_*$ will be unstable. In particular, for $\delta R(0) > 0$ the hole will extend to infinity.

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